

ON THE CONVEX PFAFF-DARBOUX THEOREM OF EKELAND AND NIRENBERG

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ABSTRACT. The classical Pfaff-Darboux Theorem, which provides local ‘normal forms’ for 1-forms on manifolds, has applications in the theory of certain economic models [3]. However, the normal forms needed in these models come with an additional requirement of *convexity*, which is not provided by the classical proofs of the Pfaff-Darboux Theorem. (The appropriate notion of ‘convexity’ is a feature of the economic model. In the simplest case, when the economic model is formulated in a domain in \mathbb{R}^n , convexity has its usual meaning.) In [4], Ekeland and Nirenberg were able to characterize necessary and sufficient conditions for a given 1-form ω to admit a convex local normal form (and to show that some earlier attempts [2, 5] at this characterization had been unsuccessful).

In this article, after providing some necessary background, I prove a strengthened and generalized convex Pfaff-Darboux Theorem, one that covers the case of a Legendrian foliation in which the notion of convexity is defined in terms of a torsion-free affine connection on the underlying manifold. (The main result in [4] concerns the case in which the affine connection is flat.)

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1. INTRODUCTION

The Pfaff-Darboux Theorem provides a local ‘normal form’ for 1-forms on manifolds, assuming that certain constant rank conditions are met. A common version¹ of this classical theorem is the following: Let ω be a smooth 1-form on an n -manifold M and suppose that there is an integer $k > 0$ such that

$$\omega \wedge (d\omega)^k \text{ vanishes identically on } M$$

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¹There are many variants. See [1, Chapter II, §3].

while

$$\omega \wedge (d\omega)^{k-1} \text{ is nowhere vanishing on } M.$$

Then each $x \in M$ has an open neighborhood $U \subset M$ on which there exist (smooth) functions $y^1, \dots, y^k, p_2, \dots, p_k$ and a nonvanishing function a such that²

$$(1.1) \quad U^*\omega = a(dy^1 + p_2 dy^2 + \dots + p_k dy^k).$$

Since

$$U^*(\omega \wedge (d\omega)^{k-1}) = (-1)^{k(k-1)/2} (k-1)! a^k dy^1 \wedge \dots \wedge dy^k \wedge dp_2 \wedge \dots \wedge dp_k,$$

the functions $y^1, \dots, y^k, p_2, \dots, p_k$ in this representation must be independent on U .

The normal form (1.1) is often written more symmetrically as

$$(1.2) \quad U^*\omega = a_1 du^1 + a_2 du^2 + \dots + a_k du^k,$$

where the a_i do not simultaneously vanish in U . In this representation, the independence of the functions $y^1, \dots, y^k, p_2, \dots, p_k$ translates into the condition that the mapping

$$(1.3) \quad (u, [a]) : U \rightarrow \mathbb{R}^k \times \mathbb{RP}^{k-1} = \mathbb{P}(T^*\mathbb{R}^k)$$

be a submersion.

In fact, the representation (1.2) is more common in treatises on mathematical economics, where the Pfaff-Darboux Theorem plays an important role [2]. However, the normal forms needed in these models sometimes come with an additional requirement of *convexity*, i.e., the underlying manifold is $M = \mathbb{R}^n$ (or an open domain in \mathbb{R}^n), and one would like to have the functions a_i be *positive* while the functions u^i are *strictly convex*, i.e., they have positive definite Hessians.³ A useful reference for the role of convexity in economic models is the book [3].

Now, it turns out that constructing such a *convex* Pfaff-Darboux representation is not always possible, which raises the question of how to determine when one exists. In [4], Ekeland and Nirenberg were able to provide necessary and sufficient conditions for a given 1-form $\omega \in \Omega^1(\mathbb{R}^n)$ to admit a local convex Pfaff-Darboux normal form. They also constructed examples that showed that some earlier attempts [2, 5] to find such conditions had been unsuccessful.

In this note, after providing some necessary background, I prove a generalization of the convex Pfaff-Darboux Theorem of Ekeland and Nirenberg. This treatment has some notable features that make it of interest for the general problem. First, the proof of Ekeland and Nirenberg does not rely on the classical Pfaff-Darboux Theorem; instead, it constructs the required convex representation directly using PDE techniques, thereby reproving the Pfaff-Darboux Theorem in this special case. The proof below assumes the classical Pfaff-Darboux Theorem, and so the argument can more directly focus on choosing a Pfaff-Darboux representation that satisfies the convexity requirements. This results in a shorter proof, one that also brings the nature of the convexity requirements more sharply into focus. Second, the notion of *strict convexity* turns out to be meaningful on any manifold endowed with a torsion-free affine connection, and the proof below covers this more general situation with no extra work. Third, the proof yields a stronger result, in that it produces

²Throughout this article, I adopt the convention that, when $L \subset M$ is a submanifold and ψ is a differential form on M , then $L^*\psi$ denotes the *pullback* of ψ to L .

³Sometimes one only requires *weak convexity*, i.e., that the Hessian of u^i be positive definite on each of its level sets.

a local convex Pfaff-Darboux representation of ω adapted to any ω -Legendrian foliation that satisfies a certain geometrically natural positivity condition, one that is equivalent to the condition of Ekeland and Nirenberg.

2. CLASSICAL PFAFF-DARBOUX THEOREMS

Let ω be a smooth 1-form on an n -manifold M that, for some integer $k > 0$, satisfies

$$(2.1) \quad \omega \wedge (d\omega)^k \text{ vanishes identically on } M$$

while

$$(2.2) \quad \omega \wedge (d\omega)^{k-1} \text{ is nowhere vanishing on } M.$$

The integer $k-1$ is known as the *Pfaff rank* of ω [1, Chapter II, §3]. Note that $k \leq \frac{1}{2}(n+1)$; when $k = \frac{1}{2}(n+1)$, ω is said to be a *contact form* on M .

When ω satisfies (2.1) and (2.2), so does $\tilde{\omega} = f\omega$ for any nonvanishing function f on M , since $\tilde{\omega} \wedge (d\tilde{\omega})^{r-1} = f^r \omega \wedge (d\omega)^{r-1}$ for all integers $r > 0$.

2.1. Canonical subbundles. An ω satisfying (2.1) and (2.2) defines a *kernel subbundle* $K = \omega^{-1}(0) \subset TM$ of corank 1 and a subbundle $A \subset K$ of corank $2(k-1)$ in K by the rule that, for each $x \in M$,

$$(2.3) \quad A_x = \{v \in K_x \mid d\omega(v, w) = 0, \forall w \in K_x\}.$$

Replacing ω by $\tilde{\omega} = f\omega$ for any nonvanishing function f does not change K or A .

When $k > 1$, then $K \subset TM$ is not an integrable plane field, but the subbundle $A \subset TM$ is always integrable, since it is the Cauchy characteristic plane field of the differential ideal \mathcal{I} generated by ω (see [1, Chapter II, Prop. 2.1]). In the contact case, i.e., when $n = 2k-1$ (which is, in some sense, generic), one has $A = (0)$.

There is a nondegenerate, skew-symmetric bilinear pairing $B_\omega : K/A \times K/A \rightarrow \mathbb{R}$ defined by

$$B_\omega(v+A_x, w+A_x) = d\omega(v, w),$$

when $v, w \in K_x$. It satisfies $B_{f\omega} = fB_\omega$ for any nonvanishing function f on M .

Note that any subspace $W \subset T_x$ on which both ω and $d\omega$ vanish, must, first of all, satisfy $W \subset K_x$ (since ω vanishes on W), and second, must have codimension at least $k-1$ in K_x , since $d\omega$, as a skew-symmetric form on K_x , has Pfaff rank $k-1$. Moreover, if W does have codimension $k-1$ in K_x , then it must contain A_x , so that W/A_x is a null subspace of B_ω .

2.2. Legendrian submanifolds and Grassmannians. Any submanifold $L \subset M$ that satisfies $L^*\omega = 0$, i.e., an *integral manifold* of ω , must also satisfy $L^*d\omega = 0$ and hence, by the above linear algebra discussion, must have codimension at least k in M . When $L \subset M$ is an integral manifold of ω of codimension k , it is said to be an *ω -Legendrian submanifold*.

In particular, if $L \subset M$ is ω -Legendrian, then, for each $x \in L$, the tangent space $T_x L$ satisfies $A_x \subset T_x L \subset K_x$ while B_ω vanishes identically on $T_x L/A_x \subset K_x/A_x$.

This motivates defining the *Legendrian Grassmannian* $\text{Leg}_x(\omega) \subset \text{Gr}^k(T_x M)$ to be the set of subspaces $W \subset K_x$ that have codimension k in $T_x M$ and on which both ω and $d\omega$ vanish. By the above remarks, it follows that $\text{Leg}_x(\omega)$ can be canonically identified with the *Lagrangian Grassmannian* $\text{Lag}(K_x/A_x) \subset \text{Gr}^{k-1}(K_x/A_x)$ consisting of the $(k-1)$ -dimensional subspaces of K_x/A_x on which B_ω vanishes.

Hence, $\text{Leg}_x(\omega)$ is naturally a smooth manifold of dimension $\frac{1}{2}k(k-1)$. Moreover, the (disjoint) union

$$\text{Leg}(\omega) = \bigcup_{x \in M} \text{Leg}_x(\omega) \subset \text{Gr}^k(TM)$$

is a smooth subbundle, and $\text{Leg}(f\omega) = \text{Leg}(\omega)$ for all nonvanishing f .

2.3. A local normal form. One version of the *Pfaff-Darboux Theorem* [1, Chapter II, Theorem 3.1] states that, when $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2), each $x \in M$ has an open neighborhood $U \subset M$ on which there exist smooth functions u^1, \dots, u^k and a_1, \dots, a_k (with not all a_i simultaneously vanishing) so that

$$(2.4) \quad U^*\omega = a_1 du^1 + \dots + a_k du^k.$$

Moreover, the mapping $(u, [a]) : U \rightarrow \mathbb{R}^k \times \mathbb{RP}^{k-1}$ is a submersion. (In fact, the kernel subbundle of the differential of this mapping is the restriction of A to U .)

Conversely, the existence of functions u^i and a_i for $1 \leq i \leq k$ on an open set $U \subset M$ satisfying (2.4) with the a_i not all simultaneously vanishing and having the property that $(u, [a]) : U \rightarrow \mathbb{R}^k \times \mathbb{RP}^{k-1}$ be a submersion implies that both (2.1) and (2.2) hold on U .

2.4. Geometry of the normal form. It will be useful to have a geometric interpretation of the Pfaff-Darboux Theorem. Now, in the representation (2.4), the functions u^i have independent differentials, i.e., $du^1 \wedge \dots \wedge du^k$ does not vanish on U . Consequently, the simultaneous level sets of the functions u^i define a foliation \mathcal{L} of $U \subset M$ by ω -Legendrian submanifolds, i.e., an ω -Legendrian foliation.

Conversely, given an ω -Legendrian foliation \mathcal{L} on an open subset $V \subset M$, each point $x \in V$ will have an open neighborhood $U \subset V$ in which the leaves of \mathcal{L} are the fibers of a submersion $u = (u^i) : U \rightarrow \mathbb{R}^k$. Since ω vanishes when pulled back any fiber of u , it follows that there exists a mapping $a = (a_i) : U \rightarrow \mathbb{R}^k$ such that $U^*\omega = a_1 du^1 + \dots + a_k du^k$.

Thus, a geometric interpretation of the Pfaff-Darboux Theorem is the statement that, when $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2), each point $x \in M$ has an open neighborhood $U \subset M$ on which there exists an ω -Legendrian foliation.

2.5. Variants and extensions. There are a number of variants and extensions of the classical Pfaff-Darboux Theorem that can all be seen to be equivalent to the above versions by elementary arguments [1, Chapter II, §3]. In this article, two such variants will be important. For convenience of reference, they will be stated as propositions.

Proposition 1. *Suppose that $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2). Then for each $x \in M$ and $W \in \text{Leg}_x(\omega)$, there exists a ω -Legendrian submanifold $L \subset M$ such that $x \in L$ and $W = T_x L$.*

Proposition 2. *Suppose that $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2) and that $L \subset M$ is an embedded ω -Legendrian submanifold. Then each $x \in L$ has an open neighborhood $U \subset M$ on which there exists an ω -Legendrian foliation \mathcal{L} with the property that $L \cap U$ is a leaf of \mathcal{L} .*

3. CONVEXITY AND AFFINE MANIFOLDS

3.1. Classical convexity. When $M = \mathbb{R}^n$, there is a notion of *strict convexity* of a function u , which is the condition that the Hessian quadratic form $H(u)$ be positive definite, where

$$(3.1) \quad H(u) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \otimes dx^j$$

and where x^1, \dots, x^n are the usual affine linear coordinates in \mathbb{R}^n . Note that strict convexity is an affine-invariant notion on \mathbb{R}^n .

Motivated by applications in economics, Ekeland and Nirenberg [4] asked what further conditions one must impose on an $\omega \in \Omega^1(\mathbb{R}^n)$ satisfying (2.1) and (2.2) in order to know that one can choose the functions u^j and a_j in the representation (2.4) so that the u^j be strictly convex and the a_j be positive. It is not hard to show, by example, that *some* further condition on ω is necessary to guarantee the existence of such a convex representation. (See the discussion at the beginning of §3.3.)

They showed that two earlier articles [2, 5] claiming to provide such necessary and sufficient conditions were flawed (indeed, they exhibited counterexamples to the claims of these articles) and then produced their own condition, which they showed to be necessary and sufficient.

In this note, I will show that their main result, properly formulated, holds good on an n -manifold M endowed with a torsion-free affine connection, not just on \mathbb{R}^n endowed with the (flat) affine connection it inherits as a vector space.

3.2. Affine connections and convexity. Let ∇ be a torsion-free affine connection on an n -manifold M^n , i.e., ∇ is a first-order, linear differential operator

$$(3.2) \quad \nabla : \Omega^1(M) \rightarrow \Omega^1(M) \otimes \Omega^1(M)$$

that obeys the Leibnitz rule

$$(3.3) \quad \nabla(f\eta) = df \otimes \eta + f \nabla(\eta)$$

for all smooth functions f on M and smooth 1-forms η on M . The assumption that ∇ be torsion-free is the condition that the associated (second-order) *Hessian operator* $H(u) = \nabla(du)$ be a symmetric $(0, 2)$ -tensor for each smooth function u on M .

A smooth function u on M is said to be *strictly ∇ -convex* if, as a quadratic form, $H(u)$ is positive definite at every point of M .

When $M = \mathbb{R}^n$ and ∇ is the standard (flat) connection, satisfying $\nabla(dx^i) = 0$ for all of the coordinate functions x^i , then $H(u)$ is the usual Hessian tensor (3.1), and this notion of convexity is simply the classical one.

In the more general case, when $x = (x^i) : U \rightarrow \mathbb{R}^n$ is a local coordinate chart, one has

$$(3.4) \quad H(x^k) = \nabla(dx^k) = \Gamma_{ij}^k dx^i \otimes dx^j$$

where $\Gamma_{ij}^k = \Gamma_{ji}^k \in C^\infty(U)$ are the coefficients of the connection ∇ relative to the coordinate chart $x = (x^i)$. The general coordinate formula for H then becomes

$$(3.5) \quad H(u) = \left(\frac{\partial^2 u}{\partial x^i \partial x^j} + \Gamma_{ij}^k \frac{\partial u}{\partial x^k} \right) dx^i \otimes dx^j.$$

Thus, ∇ -convexity of u is expressible in terms of a condition on the 2-jet of u , slightly more general than the condition for classical convexity.

Adopting the usual conventions

$$(3.6) \quad \begin{aligned} \alpha \wedge \beta &= \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha) \\ \alpha \circ \beta &= \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha), \end{aligned}$$

one sees that, for a 1-form ω of the form

$$(3.7) \quad \omega = a_1 du^1 + \cdots + a_k du^k,$$

one has (using the summation convention)

$$(3.8) \quad \begin{aligned} \nabla \omega &= da_i \otimes du^i + a_i H(u^i) \\ &= da_i \wedge du^i + da_i \circ du^i + a_i H(u^i) \\ &= d\omega + S\omega, \end{aligned}$$

where I have introduced the notation $S\omega$ to denote the *symmetrization* of $\nabla(\omega)$. Thus, $S\omega = \nabla\omega - d\omega$ is a well-defined quadratic form on M . (Of course, the linear, first-order differential operator S depends on ∇ .)

3.3. A positivity condition. If, in a local representation (3.7), the a_i are positive and the functions u^j are strictly ∇ -convex, then the quadratic form

$$S\omega = da_i \circ du^i + a_i H(u^i)$$

is strictly positive definite on the plane field $N \subset TM$ defined by $du^1 = du^2 = \cdots = du^k = 0$ since, on N , the terms $da_i \circ du^i$ in $S\omega$ vanish and one is left with the positive definite expression $a_i H(u^i)$. (Of course, both ω and $d\omega$ vanish when pulled back to the plane field N .)

Thus, one sees that the ω -Legendrian foliation \mathcal{L} defined in U by $du^1 = du^2 = \cdots = du^k = 0$ has the property that the quadratic form $S\omega$ is positive definite on each of the leaves of \mathcal{L} . It turns out that this necessary condition for a local ‘convex’ Pfaff-Darboux representation compatible with the ω -Legendrian foliation \mathcal{L} is also sufficient.⁴

Theorem 1. *Suppose ∇ be a torsion-free affine connection on M , that $\omega \in \Omega^1(M)$ satisfy (2.1) and (2.2) for some $k > 0$, and that \mathcal{L} be an ω -Legendrian foliation on M with the property that $S\omega$ pulls back to each leaf of \mathcal{L} to be positive definite. Then each $x \in M$ has an open neighborhood $U \subset M$ on which there exist strictly ∇ -convex functions u^1, \dots, u^k that are constant on the leaves of \mathcal{L} in U and positive functions a_1, \dots, a_k such that*

$$(3.9) \quad U^*\omega = a_1 du^1 + \cdots + a_k du^k.$$

Before giving the proof of Theorem 1, I will state one of its corollaries, so that it can be compared with the main result of Ekeland and Nirenberg [4, Theorem 1].

First, some useful terminology. As always, assume that ω satisfies (2.1) and (2.2) for some $k > 0$.

Definition 1. An ω -Legendrian subspace $W \subset T_x M$ is ∇ -positive for ω if the restriction of the quadratic form $S\omega$ to W is positive definite.

⁴It is worth pointing out that the same conclusion about the positive definiteness of $S\omega$ on the leaves of \mathcal{L} would have followed if one had merely assumed that each u^i be only ‘strictly ∇ -quasi-convex’, i.e., that du^i be nonvanishing and $H(u^i)$ be positive definite when restricted to the hyperplane field $du^i = 0$. Compare [4, Lemma 1], and the preceding discussion about their Problem 2.

Let $\text{Leg}^+(\omega, \nabla) \subseteq \text{Leg}(\omega)$ denote the set of ω -Legendrian subspaces that are ∇ -positive for ω . Then $\text{Leg}^+(\omega, \nabla)$ is a (possibly empty) open subset of $\text{Leg}(\omega)$. Consequently, the set of points $x \in M$ for which there exists a ∇ -positive, ω -Legendrian subspace $W \subset T_x M$ is an open subset of M . Also, note that, since such a W contains A_x , it follows that $S\omega$ must be positive definite on A_x .

Corollary 1. *Suppose that ∇ be a torsion-free affine connection on M , that $\omega \in \Omega^1(M)$ satisfy (2.1) and (2.2) for some $k > 0$, and that there exist a $W \in \text{Leg}^+(\omega, \nabla)$ with $W \subset T_x M$. Then $x \in M$ has an open neighborhood $U \subset M$ on which there exist strictly ∇ -convex functions u^1, \dots, u^k and positive functions a_1, \dots, a_k such that*

$$(3.10) \quad U^* \omega = a_1 du^1 + \dots + a_k du^k.$$

The proof of Corollary 1 follows by applying Propositions 1 and 2 to produce an ω -Legendrian foliation \mathcal{L} on an open neighborhood V of x whose leaf through x has W as a tangent space. Since $S\omega$ is positive definite on W , it follows that it is positive definite on all the tangent spaces to the leaves of \mathcal{L} in some (possibly) smaller x -neighborhood $V' \subset V$. Now apply Theorem 1 to \mathcal{L} on V' .

Remark 1. In the special case in which $M = \mathbb{R}^n$ and ∇ is the flat connection satisfying $\nabla(dx^i) = 0$ for x^i the standard coordinates on \mathbb{R}^n , Corollary 1 simply becomes Theorem 1 of Ekeland and Nirenberg [4], since their Condition 3 turns out to be equivalent to the existence of a $W \in \text{Leg}^+(\omega, \nabla)$ with $W \subset T_x M$ in this case.

Proof of Theorem 1. There exists an x -neighborhood $V_0 \subset M$ on which there exist smooth functions y^1, \dots, y^k vanishing at x so that the leaves of $dy^1 = \dots = dy^k = 0$ are intersections of the leaves of \mathcal{L} with V_0 as well as functions p_2, \dots, p_k , also vanishing at x , and a nonvanishing function a so that

$$V_0^* \omega = a(dy^1 + p_2 dy^2 + \dots + p_k dy^k)$$

By reversing the signs of a and the y^i , if necessary, one can assume that $a(x) > 0$. Let $W \subset T_x M$ be the tangent to the leaf of \mathcal{L} that passes through x , so that W is the common kernel of the dy^i evaluated at x .

Set $\bar{\omega} = a^{-1}\omega$ and note that, since $d\bar{\omega} \equiv a^{-1}d\omega \bmod \omega$, it follows that \mathcal{L} is also $\bar{\omega}$ -Legendrian. Moreover, since

$$S\bar{\omega} = d(a^{-1}) \circ \omega + a^{-1}S\omega,$$

it follows that the tangent spaces of \mathcal{L} (which, of course, satisfy $\omega = 0$) are also ∇ -positive for $\bar{\omega}$. Since $\omega = a\bar{\omega}$ and $a > 0$, finding the desired convex representation for $\bar{\omega}$ will also yield one for ω . Thus, it suffices to prove the theorem with $\bar{\omega}$ in the place of ω , i.e., to assume that $a = 1$, so I will do that from now on. Thus,

$$V_0^* \omega = dy^1 + p_2 dy^2 + \dots + p_k dy^k.$$

Since $\omega \wedge (d\omega)^{k-1} \neq 0$, the functions $y^1, \dots, y^k, p_2, \dots, p_k$ have linearly independent differentials at x .

Restricting to V_0 , i.e., setting $M = V_0$, one has

$$S\omega = H(y^1) + dp_2 \circ dy^2 + \dots + dp_k \circ dy^k + p_2 H(y^2) + \dots + p_k H(y^k).$$

Since the p_j vanish at x , it follows that, when restricted to $W \subset T_x M$, the two quadratic forms $H(y^1)$ and $S\omega$ are equal. Thus $H(y^1)$ is positive definite on W ,

and so there is a constant $c > 0$ so that $H(y^1) + c(dy^2)^2 + \cdots + c(dy^k)^2$ is positive definite on $K_x = \{v \in T_x M \mid dy^1(v) = 0\}$. Writing

$$\omega = d(y^1 + \frac{1}{2}c(y^2)^2 + \cdots + \frac{1}{2}c(y^k)^2) + (p_2 - cy^2)dy^2 + \cdots + (p_k - cy^k)dy^k$$

and observing that

$$\begin{aligned} H(y^1 + \frac{1}{2}c(y^2)^2 + \cdots + \frac{1}{2}c(y^k)^2) &= H(y^1) + c(dy^2)^2 + \cdots + c(dy^k)^2 \\ &\quad + cy^2 H(y^2) + \cdots + cy^k H(y^k) \end{aligned}$$

shows that, setting

$$\bar{y}^1 = y^1 + \frac{1}{2}c(y^2)^2 + \cdots + \frac{1}{2}c(y^k)^2, \quad \bar{y}^i = y^i, \quad \text{and} \quad \bar{p}_i = p_i - cy^i,$$

one has $\omega = d\bar{y}^1 + \bar{p}_2 d\bar{y}^2 + \cdots + \bar{p}_k d\bar{y}^k$.

Thus, one could have chosen the functions $y^1, \dots, y^k, p_2, \dots, p_k$ with $H(y^1)$ being positive definite on the hyperplane K_x . Assume now that this was done.

It still needs to be shown that one can choose the functions $y^1, \dots, y^k, p_2, \dots, p_k$ with $H(y^1)$ being positive definite on all of $T_x M$, not just on K_x , which is the kernel of dy^1 at x . To do this, note that, if ϕ is any smooth function on a neighborhood of the origin in \mathbb{R} , then

$$H(\phi(y^1)) = \phi'(y^1) H(y^1) + \phi''(y^1) (dy^1)^2.$$

Hence, by choosing a ϕ with $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi''(0) > 0$ sufficiently large, I can arrange that $\phi(y^1)$ be strictly ∇ -convex at x . Since

$$\omega = \frac{1}{\phi'(y_1)} (d(\phi(y^1)) + \phi'(y_1)p_2 dy^2 + \cdots + \phi'(y_1)p_k dy^k),$$

one sees that the functions $\bar{y}^1, \dots, \bar{y}^k, \bar{p}_2, \dots, \bar{p}_k$, where

$$\bar{y}^1 = \phi(y^1), \quad \text{and} \quad \bar{y}^i = y^i, \quad \bar{p}_i = \phi'(y_1)p_i, \quad 2 \leq i \leq k,$$

(with $a = 1/\phi'(y_1) > 0$), give a Pfaff-Darboux representation for ω that is compatible with the foliation \mathcal{L} and for which \bar{y}^1 is strictly ∇ -convex.⁵

Thus, one can assume henceforth that, on an open x -neighborhood $V_1 \subset M$, one has a representation of the form

$$\omega = dy^1 + p_2 dy^2 + \cdots + p_k dy^k,$$

where the functions $y^1, \dots, y^k, p_2, \dots, p_k \in C^\infty(V_1)$ all vanish at x , the equations $dy^i = 0$ define the tangents to the leaves of \mathcal{L} in V_1 , and y^1 is strictly ∇ -convex.

Under these assumptions, there is a constant $b > 0$ sufficiently large so that $H(y^i + by^1) = H(y^i) + bH(y^1)$ is positive definite at x for $2 \leq i \leq k$. Thus, writing

$$\omega = (1 - b(p_2 + \cdots + p_k)) dy^1 + p_2 d(y^2 + by^1) + \cdots + p_k d(y^k + by^1),$$

it follows that I can, after restricting to an x -neighborhood $V_2 \subset V_1$ on which the function $a = (1 - b(p_2 + \cdots + p_k))$ is positive, dividing by $a > 0$, and replacing y^j by $y^j + by^1$ and p_j by p_j/a for $2 \leq j \leq k$, assume that I have a representation

$$\omega = dy^1 + p_2 dy^2 + \cdots + p_k dy^k,$$

in which all of the $H(y^j)$ are positive definite at x , i.e., the y^j are strictly ∇ -convex on some neighborhood of x and the p_i all vanish at x .

⁵This is the same idea that Ekeland and Nirenberg used in their generalization of their Theorem 1 to cover the quasi-convex case.

Finally, for $\varepsilon > 0$ and sufficiently small, write

$$\omega = d(y^1 - \varepsilon(y^2 + \cdots + y^k)) + (p_2 + \varepsilon) dy^2 + \cdots + (p_k + \varepsilon) dy^k.$$

Then, setting $u^1 = y^1 - \varepsilon(y^2 + \cdots + y^k)$ and $u^j = y^j$ for $j > 1$ and setting $a_1 = 1$ and $a_j = \varepsilon + p_j$ for $j > 1$, one achieves the desired convex Pfaff-Darboux representation on an open x -neighborhood $U \subset V_2$. \square

Remark 2 (Global considerations). While Theorem 1 gives necessary and sufficient conditions for the existence of local ∇ -convex Pfaff-Darboux representations, for applications one would like to know something about how large an open set in the model M one can cover with such a representation, and this seems to be a subtle problem.

Even in the simplest case of a 3-manifold M endowed with a contact 1-form ω and a torsion-free affine connection ∇ for which $S\omega$ is positive definite on the 2-plane bundle $K \subset TM$, it is not clear how to characterize the domains $U \subset M$ that support a ∇ -convex Pfaff-Darboux representation. It is clear that such a U must be ω -tight, but this does not appear to be sufficient.

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